Looping the loop

Entering a vertical loop, gravity will slow the carriage as it rises; we are again exchanging kinetic energy for gravitational potential energy. Unlike the horizontal case, the velocity v is not uniform, even when the radius of curvature r is constant. From the conservation of energy, we learn something interesting. We know that the carriage will rise (or fall) a height equal to twice the radius, r so:

$$\frac{1}{2}mv_{base}^2 - \frac{1}{2}mv_{top}^2 = mg\,2r$$

$$\Rightarrow \quad \frac{v_{base}^2}{r} - \frac{v_{top}^2}{r} = 4g$$

Centripetal acceleration at the base of a circular loop will *always* be exactly $4g$ greater than that at the top, no matter what values we choose for v_{base} and r. Inward acceleration at the top must never fall below $1g$ or the carriage will fall from the rails under gravity. The situation becomes worse when we remember that riders at the bottom already endure $1g$, due to gravity, before we push them up. Upward acceleration at the base will thus be at least $6g$, which is too high to be either comfortable or safe for most people. If a downward loop were executed, it would be even more unhealthy.

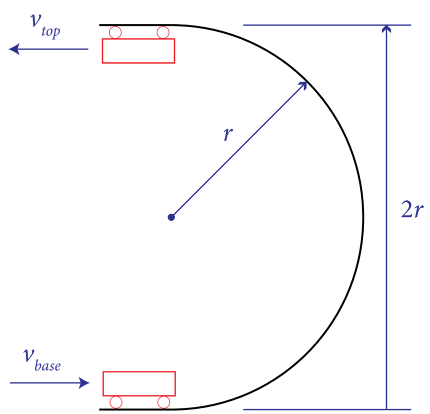

Figure 1.8 A vertical half-loop.

We need to establish two distinct inequalities:

$$\frac{v_{top}^2}{r} \geq 1g \qquad \frac{v_{base}^2}{r} \leq ng$$

where n is the g limit applied to a healthy human (assuming unhealthy ones do not ride).

We might design a loop with centripetal acceleration at the top close to (but not less than) $1g$, which will give a *perception* of "zero g", as experienced by astronauts in orbit.

We're used to being tugged and *not* falling.

From the above, we can obtain a joint inequality:

$$\frac{v_{base}^2}{r} - \frac{v_{top}^2}{r} \leq (n-1)g$$

Substituting the earlier relation, leaves us with what we have already determined verbally, that the smallest value of n would be five, giving a total acceleration experienced at the base of $6g$.

The only answer is to vary the radius of curvature.

If we increase the radius of the lower quadrant (quarter loop), both on the rise and the fall, the centripetal acceleration can be reduced to something more tolerable — say $4g$, giving a total of $5g$, which is acceptable for the few seconds required, and will add to the thrill of the ride. However, we must also take care not to increase the height of the loop, which would add to the energy lost and so to the speed needed on entry, and the consequent centripetal acceleration. A *decrease* in the radius of the upper quadrant can compensate for the increase in that of the lower one, leaving the height unchanged, and can also raise the centripetal acceleration at the top. This further reduces the velocity, and thus acceleration, needed at the base.

The result is a "teardrop shape", familiar to riders of modern coasters.

The first equation above still holds, subject to:

$$r_{base} + r_{top} = 2r$$

We can capture the change via a parameter α:

$$r_{base} = \alpha r \qquad 1 < \alpha < 2$$
$$r_{top} = (2-\alpha)r$$

The second equation is revised, taking the equalities only:

$$\frac{v_{base}^2}{r} - \frac{v_{top}^2}{r} = \left[(n+1)\alpha - 2\right]g$$

Combining both equations, we get:

$$(n+1)\alpha - 2 = 4$$

$$\Rightarrow \quad \alpha = \frac{6}{n+1}$$

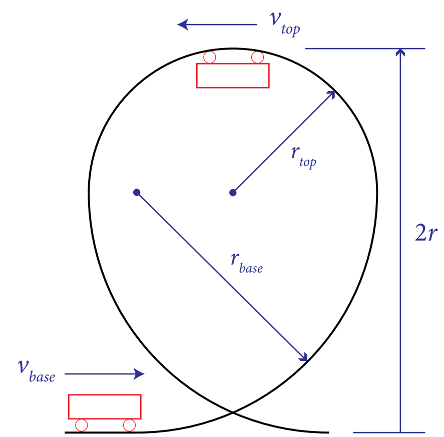

Figure 1.9 A vertical tear-drop.

So, for example, if we choose a base centripetal acceleration of $4g$, we need to multiply the base radius by 1.2, giving a ratio of 3 : 2 between that and the top radius. The total experienced would then be $5g$, allowing for gravity's contribution, which should be acceptable for the few seconds required.

Application

1.14 Making sensible assumptions for all parameters, design yourself a life-size roller coaster, complete with drops, turns and loops.

Remember, a vertical loop can be upward, with the carriage upright, or downward, with the carriage inverted. It might even twist half way. A half upward loop might precede a drop, and so on.